Conformal Properties
of
Chern-Simons Vortices
in
External Fields

C. DUVAL $(^1)$ P. A. HORVÁTHY $(^2)$ L. PALLA $(^3)$

Abstract.

The construction and the symmetries of Chern-Simons vortices in harmonic and uniform magnetic force backgrounds found by Ezawa, Hotta and Iwazaki, and by Jackiw and Pi are generalized using the non-relativistic Kaluza-Klein-type framework presented in our previous paper. All Schrödinger-symmetric backgrounds are determined.

$(^1)$ and Département de Physique, Université d'Aix-Marseille II, UFR de Luminy, Case 907, F–13288 MARSEILLE Cedex 09 (France), e-mail : duval@cpt.univ-mrs.fr.

$(^2)$ Permanent address : Département de Mathématiques, Université de Tours, Parc de Grandmont, F–37200 TOURS (France), e-mail : horvathy@univ-tours.fr

$(^3)$ Permanent address : Institute for Theoretical Physics, Eötvös University, H–1088 BUDAPEST, Puskin u. 5-7 (Hungary), e-mail : palla@ludens.elte.hu

1. Introduction

The construction of static, non-relativistic Chern-Simons solitons [1] was recently generalized to time-dependent solutions, yielding vortices in a constant external magnetic field, $\mathcal{B}$ [2-4]. Putting $\omega = \mathcal{B}/2$, the equation to be solved is

$$(1.1) \qquad i\left(D_\omega\right)_t \Psi_\omega = \left\{ -\frac{1}{2}\vec{D}_\omega^2 - \Lambda\,\Psi_\omega^*\Psi_\omega \right\}\Psi_\omega.$$

(We use units where $e = m = 1$). Here the covariant derivative means

$$(1.2) \qquad \left(D_\omega\right)_\alpha = \partial_\alpha - i(A_\omega)_\alpha - i\mathcal{A}_\alpha$$

($\alpha = 0, 1, 2$), where $\mathcal{A}_\alpha$ is a vector potential for the constant magnetic field, $\mathcal{A}_0 = 0$, $\mathcal{A}_i = \frac{1}{2}\epsilon_{ij}x^j\mathcal{B} \equiv \omega\epsilon_{ij}x^j$ ($i, j = 1, 2$) and $(A_\omega)_\alpha$ is the vector potential of Chern-Simons electromagnetism i.e. its field strength is required to satisfy the field-current identity

$$(1.3) \qquad B_\omega \equiv \epsilon^{ij}\partial_i A_\omega^j = -\frac{1}{\kappa}\rho_\omega \qquad \text{and} \qquad E_\omega^i \equiv -\partial_i A_\omega^0 - \partial_t A_\omega^i = \frac{1}{\kappa}\epsilon^{ij}J_\omega^j$$

with $\rho_\omega = \Psi_\omega^*\Psi_\omega$ and $\vec{J}_\omega = (1/2i)[\Psi^*\vec{D}_\omega\Psi_\omega - \Psi_\omega(\vec{D}_\omega\Psi_\omega)^*]$. These equations can be solved [2-4] by applying a coordinate transformation to a solution, Ψ, of the problem with $\omega = 0$ studied in Ref. [1], according to

$$(1.4) \qquad \begin{aligned} \Psi_\omega(t, \vec{x}) &= \frac{1}{\cos\omega t}\,\exp\left\{-i\omega\frac{r^2}{2}\tan\omega t\right\}\exp\left\{i\frac{\mathcal{N}}{2\pi\kappa}\omega t\right\}\Psi(\vec{X}, T), \\[2mm] (A_\omega)_\alpha &= A_\beta\frac{\partial X^\beta}{\partial x^\alpha} - \partial_\alpha\left(\frac{\omega}{2\pi\kappa}\mathcal{N}t\right), \end{aligned}$$

with

$$(1.5) \qquad T = \frac{\tan\omega t}{\omega}, \qquad \vec{X} = \frac{1}{\cos\omega t}R(\omega t)\,\vec{x}.$$

Here $\mathcal{N} = \int\Psi^*\Psi\,d^2\vec{x}$ is the vortex number and $R(\theta)$ is the matrix of a rotation by angle θ in the plane. (The pre-factor $\exp[i\mathcal{N}\omega t/2\pi\kappa]$ and the extra term $-\partial_\alpha[(\omega/2\pi\kappa)\mathcal{N}t]$ are absent from the corresponding formula of Ezawa et al. [2]). A similar construction works in a harmonic background [4].

In a previous paper [5] non-relativistic Chern-Simons theory in $2+1$ dimensions was obtained by reduction from an appropriate $(3+1)$-dimensional Lorentz manifold. As an application, we reproduced the results in Ref. [1]. Here we show that the above generalizations arise by reduction from suitable curved spaces, that they all share the (extended) Schrödinger symmetry of the model in [1], and we determine all background fields which have this property.

2. Chern-Simons theory in Bargmann space

$(2 + 1)$-dimensional non-relativistic Chern-Simons theory can be lifted to 'Bargmann space' i.e. to a 4-dimensional Lorentz manifold (M, g) endowed with a covariantly constant null vector ξ [5]. The theory is described by a massless non-linear wave equation

$$(2.1) \qquad \left\{ D_\mu D^\mu - \frac{R}{6} + \lambda \psi^* \psi \right\} \psi = 0,$$

where $D_\mu = \nabla_\mu - i a_\mu$ $(\mu = 0, 1, 2, 3)$, ∇ is the metric-covariant derivative and R denotes the scalar curvature. The scalar field ψ and the 'electromagnetic' field strength, $f_{\mu\nu} = 2\partial_{[\mu} a_{\nu]}$, are related by the identity

$$(2.2) \qquad \kappa f_{\mu\nu} = \sqrt{-g}\, \epsilon_{\mu\nu\rho\sigma} \xi^\rho j^\sigma, \qquad j^\mu = \frac{1}{2i} \left[\psi^* (D^\mu \psi) - \psi (D^\mu \psi)^* \right].$$

Eq. (2.1) [but *not* (2.2)] can be obtained from variation of the 'partial' action

$$(2.3) \qquad S = \frac{1}{2} \int_M \left\{ (D_\mu \psi)^* D^\mu \psi + \frac{R}{6} |\psi|^2 - \frac{\lambda}{2} |\psi|^4 \right\} \sqrt{-g}\, d^4x.$$

The quotient of M by the integral curves of ξ is non-relativistic space-time we denote by Q. A Bargmann space admits local coordinates $(t, \vec{x}, s)$ such that $(t, \vec{x})$ label Q and $\xi = \partial_s$. When supplemented by the equivariance condition

$$(2.4) \qquad \xi^\mu D_\mu \psi = i\psi,$$

our theory projects to a non-relativistic non-linear Schrödinger/Chern-Simons theory on the $(2 + 1)$-dimensional manifold Q. The field strength $f_{\mu\nu}$ is clearly the lift of a closed two-form $F_{\mu\nu}$ on Q. So, the vector potential may be chosen as $a_\mu = (A_\alpha, 0)$ with A_α s-independent. In this gauge, $\Psi(t, \vec{x}) = e^{-is} \psi(t, \vec{x})$ is then a function on Q.

A *symmetry* is a transformation of M which interchanges the solutions of the coupled system. Each ξ-preserving conformal transformation is a symmetry and the variational derivative $\vartheta_{\mu\nu} = 2\delta S / \delta g^{\mu\nu}$ provides us with a conserved, traceless and symmetric energy-momentum tensor. The version of Noether's theorem proved in [5] says that for any ξ-preserving conformal vectorfield (X^μ) on Bargmann space, the quantity

$$(2.5) \qquad Q_X = \int_{\Sigma_t} \vartheta_{\mu\nu} X^\mu \xi^\nu \sqrt{\gamma}\, d^2\vec{x}$$

(where the 'transverse space' Σ_t is a space-like 2-surface $t = \text{const.}$ and γ_{ij} is the metric induced on it by $g_{\mu\nu}$) is a constant of the motion. The conserved quantities are conveniently calculated using the formula [5]

$$(2.6) \qquad \vartheta_{\mu\nu} \xi^\nu = \frac{1}{2i} \left[\psi^* (D_\mu \psi) - \psi (D_\mu \psi)^* \right] - \frac{1}{6} \xi_\mu \left(\frac{R}{6} |\psi|^2 + (D^\nu \psi)^* D_\nu \psi + \frac{\lambda}{2} |\psi|^4 \right).$$

For example, M can be flat Minkowski space with metric $d\vec{X}^2 + 2dTdS$, where $\vec{X} \in \mathbf{R}^2$ and S and T are light-cone coordinates. This is the Bargmann space of a free, non-relativistic particle [6]. The system of equations (2.1,2,4) projects in this case to that of Ref. [1]; the ξ-preserving conformal transformations form the (extended) planar Schrödinger group, consisting of the Galilei group with generators $\mathcal{J}$ (rotation), $\mathcal{H}$ (time translation), $\vec{\mathcal{G}}$ (boosts), $\vec{\mathcal{P}}$ (space translations), augmented with dilatation, $\mathcal{D}$, and expansion, $\mathcal{K}$, and centrally extended by 'vertical' translation, $\mathcal{N}$. With a slight abuse of notation, the associated conserved quantities are denoted by the same symbols. (Explicit formulæ are listed in [1] and [5]). Applying any symmetry transformation to a solution of the field equations yields another solution. For example, a boost or an expansion applied to the static solution $\Psi_0(\vec{X})$ of Jackiw and Pi produces time-dependent solutions. Using the formulæ in [6] we find

$$(2.7) \qquad \Psi(T, \vec{X}) = \frac{1}{1 - kT} \exp\left\{ -\frac{i}{2}\left[2\vec{X} \cdot \vec{b} + T\vec{b}^2 + k\frac{(\vec{X} + \vec{b}T)^2}{1 - kT} \right] \right\} \Psi_0\left(\frac{\vec{X} + \vec{b}T}{1 - kT}\right),$$

which is the same as in [1].

Now we present some new results. The most general 'Bargmann' manifold was found long ago by Brinkmann [7]:

$$(2.8) \qquad g_{ij}dx^i dx^j + 2dt\left[ds + \vec{\mathcal{A}} \cdot d\vec{x}\right] + 2\mathcal{A}_0 dt^2, \qquad \mathcal{A}_0 = -U$$

where the 'transverse' metric g_{ij} as well as the 'vector potential' $\vec{\mathcal{A}}$ and the 'scalar' potential U are functions of t and $\vec{x}$ only. Clearly, $\xi = \partial_s$ is a covariantly constant null vector. The null geodesics of this metric describe particle motion in curved transverse space in an external electromagnetic fields $\vec{\mathcal{E}} = -\partial_t \vec{\mathcal{A}} - \vec{\nabla}U$ and $\mathcal{B} = \vec{\nabla} \times \vec{\mathcal{A}}$ [6].

Consider now a Chern-Simons vector potential $(a_\omega)_\mu = \left((A_\omega)_\alpha, 0\right)$ in the background (2.8). [The subscript $(\cdot)_\omega$ refers to an external-field problem]. Using that the only non-vanishing components of the metric (2.8) are g^{ij}, $g^{is} = -A^i$, $g^{ss} = 2U + A_i A^i$, $g^{st} = 1$, we find that the integrand in the partial action (2.3) is

$$(2.9) \qquad \left(\vec{D}_\omega \psi\right)^* \cdot \vec{D}_\omega \psi + i\left[\left((D_\omega)_t \psi\right)^* \psi - \psi^*(D_\omega)_t \psi\right] + \frac{R}{6}|\psi|^2 - \frac{\lambda}{2}|\psi|^4,$$

where the covariant derivative $(D_\omega)_\alpha$ means (1.2) with vector potential $\mathcal{A}_\alpha$. Thus, including the 'vector-potential' components into the metric (2.8) results, after reduction, simply in modifying the covariant derivative D_α in 'empty' space ($\mathcal{A}_\alpha = 0$) according to $D_\alpha \rightarrow (D_\omega)_\alpha$. The associated equation of motion is hence Eq. (1.1). (This conclusion can also be reached directly by studying the wave equation (2.1)).

Let now φ denote a conformal Bargmann diffeomorphism between two Bargmann spaces i.e. let $\varphi : (M, g, \xi) \rightarrow (M', g', \xi')$ be such that $\varphi^\star g' = \Omega^2 g$ and $\xi' = \varphi_\star \xi$. Such a

mapping projects to a diffeomorphism of the quotients, Q and Q' we denote by Φ. Then the same proof as in Ref. [5] allows one to show that if (a'_μ, ψ') is a solution of the field equations on M', then

$$(2.10) \qquad a_\mu = (\varphi^\star a')_\mu \qquad \text{and} \qquad \psi = \Omega\, \varphi^\star \psi'$$

is a solution of the analogous equations on M. Locally we have $\varphi(t, \vec{x}, s) = (t', \vec{x}', s')$ with $(t', \vec{x}') = \Phi(t, \vec{x})$ and $s' = s + \Sigma(t, \vec{x})$ so that $\psi = \Omega\, \varphi^\star \psi'$ reduces to

$$(2.11) \qquad \Psi(t, \vec{x}) = \Omega(t)\, e^{i\Sigma(t,\vec{x})} \Psi'(t', \vec{x}'), \qquad A_\alpha = \Phi^\star A'_\alpha \qquad (\alpha = 0, 1, 2).$$

Note that φ takes a ξ-preserving conformal transformation of (M, g, ξ) into a ξ'-preserving conformal transformation of (M', g', ξ'). Conformally related Bargmann spaces have therefore isomorphic symmetry groups.

The associated conserved quantities can be related by comparing the expressions in Eq. (2.6). Note first that, for ψ as in Eq. (2.10), $D_\mu \psi = \Omega\,(\varphi^\star D'_\mu \psi') + \Omega^{-1} \nabla'_\nu \Omega\, \varphi^\star \psi'$. Using $R = \Omega^2\, \varphi^\star R' + 6\Omega^{-1} \nabla'_\nu \nabla'^\nu \Omega$ and $\xi_\mu = \Omega^{-2} \xi'_\mu$ as well as $\Omega = \Omega(t)$ and that $g^{\mu t}$ is non-vanishing only for $\mu = s$ one finds hence that $\vartheta_{\mu\nu} \xi^\nu = \Omega^2 \varphi^\star (\vartheta'_{\mu\nu} \xi'^\nu)$. But $\sqrt{\gamma} = \Omega^{-2} \varphi^\star \sqrt{\gamma'}$. Therefore, the conserved quantity (2.5) associated to $X = (X^\mu)$ on (M, g, ξ) and to $X' = \varphi_\star X$ on (M', g', ξ') coincide,

$$(2.12) \qquad Q_X = \varphi^\star Q'_{X'}.$$

The labels of the generators are, however, different (see the examples below).

3. Flat examples

Consider now the Lorentz metric

$$(3.1) \qquad d\vec{x}_{osc}^2 + 2dt_{osc} ds_{osc} - \omega^2 r_{osc}^2 dt_{osc}^2$$

where $\vec{x}_{osc} \in \mathbf{R}^2$, $r_{osc} = |\vec{x}_{osc}|$ and ω is a constant. Its null geodesics correspond to a non-relativistic, spinless particle in an oscillator background [6,9]. Requiring equivariance (2.4), the wave equation (2.1) reduces to

$$(3.2) \qquad i\partial_{t_{osc}} \Psi_{osc} = \left\{ -\frac{\vec{D}^2}{2} + \frac{\omega^2}{2} r_{osc}^2 - \Lambda\, \Psi_{osc} \Psi_{osc}^* \right\} \Psi_{osc}$$

$(\vec{D} = \vec{\partial} - i\vec{A}, \Lambda = \lambda/2)$, which describes Chern-Simons vortices in a harmonic force background, studied in Ref. [3]. The clue is that the mapping $\varphi(t_{osc}, \vec{x}_{osc}, s_{osc}) = (T, \vec{X}, S)$ [9], where

$$(3.3) \qquad T = \frac{\tan \omega\, t_{osc}}{\omega}, \qquad \vec{X} = \frac{\vec{x}_{osc}}{\cos \omega t_{osc}}, \qquad S = s_{osc} - \frac{\omega r_{osc}^2}{2} \tan \omega t_{osc}$$

carries the oscillator metric (3.1) conformally into the free form, $d\vec{X}^2 + 2dTdS$, with conformal factor $\Omega(t_{\text{osc}}) = |\cos \omega t_{\text{osc}}|^{-1}$ such that $\varphi_\star \partial_{s_{\text{osc}}} = \partial_S$. Our formula lifts the coordinate transformation of Ref. [4] to Bargmann space.

A solution in the harmonic background can be obtained by Eq. (2.11). A subtlety arises, though: the mapping (3.3) is many-to-one: it maps each 'open strip'

$$(3.4) \qquad\qquad I_j = \big\{ (\vec{x}_{\text{osc}}, t_{\text{osc}}, s_{\text{osc}}) \,\big|\, (j - \tfrac{1}{2})\pi < \omega t_{\text{osc}} < (j + \tfrac{1}{2})\pi \big\},$$

where $j = 0, \pm 1, \ldots$ corresponding to a half oscillator-period onto the full Minkowski space. Application of (2.11) with Ψ an 'empty-space' solution yields, in each I_j, a solution, $\Psi_{\text{osc}}^{(j)}$. However, at the contact points $t_j \equiv (j + 1/2)(\pi/\omega)$, these fields may not match. For example, for the 'empty-space' solution obtained by an expansion, Eq. (2.7) with $\vec{b} = 0$, $k \neq 0$,

$$(3.5) \qquad \lim_{t_{\text{osc}} \to t_j - 0} \Psi_{\text{osc}}^{(j)} = (-1)^{j+1} \frac{\omega}{k} e^{-i\frac{\omega^2}{2k} r_{\text{osc}}^2} \Psi_0\big(-\frac{\omega}{k}\vec{x}\big) = - \lim_{t_{\text{osc}} \to t_j + 0} \Psi_{\text{osc}}^{(j+1)}.$$

Then continuity is restored by including the 'Maslov' phase correction [10]:

$$(3.6) \qquad \begin{cases} \Psi_{\text{osc}}(t_{\text{osc}}, \vec{x}_{\text{osc}}) = (-1)^j \dfrac{1}{\cos \omega t_{\text{osc}}} \exp\left\{ -\dfrac{i\omega}{2} r_{\text{osc}}^2 \tan \omega t_{\text{osc}} \right\} \Psi(T, \vec{X}) \\[3mm] (A_{\text{osc}})_0(t_{\text{osc}}, \vec{x}_{\text{osc}}) = \dfrac{1}{\cos^2 \omega t_{\text{osc}}} \big[A_0(T, \vec{X}) - \omega \sin \omega t_{\text{osc}} \, \vec{x}_{\text{osc}} \cdot \vec{A}(T, \vec{X}) \big], \\[3mm] \vec{A}_{\text{osc}}(t_{\text{osc}}, \vec{x}_{\text{osc}}) = \dfrac{1}{\cos \omega t_{\text{osc}}} \vec{A}(T, \vec{X}), \end{cases}$$

where j is as in (3.4). Eq. (3.6) extends the result in Ref. [4] from $|t_{\text{osc}}| < \pi/2\omega$ to any t_{osc}. For the static solution in [1] or for that obtained from it by a boost, $\lim_{t_{\text{osc}} \to t_j} \Psi_{\text{osc}}^{(j)} = 0$ and the inclusion of the correction factor is not mandatory.

Chern-Simons theory in the oscillator-metric has again a Schrödinger symmetry, whose generators are related to those in 'empty' space as

$$(3.7) \qquad \begin{cases} J_{\text{osc}} = \mathcal{J}, \\ H_{\text{osc}} = \mathcal{H} + \omega^2 \mathcal{K}, \\ (C_{\text{osc}})_\pm = \big(\mathcal{H} - \omega^2 \mathcal{K} \pm 2i\omega \mathcal{D} \big), \\ (\vec{P}_{\text{osc}})_\pm = \big(\vec{\mathcal{P}} \pm i\omega \vec{\mathcal{G}} \big), \\ N_{\text{osc}} = \mathcal{N}. \end{cases}$$

The oscillator-Hamiltonian, H_{osc}, is hence a combination of the Hamiltonian and of the expansion valid for $\omega = 0$, etc. The generators H_{osc} and $(C_{\text{osc}})_\pm$ span $o(2,1)$ and the

$(\vec{P}_{\text{osc}})_{\pm}$ generate the two-dimensional Heisenberg algebra [9]. Eq. (3.7) adds $(C_{\text{osc}})_{\pm}$ and $(\vec{P}_{\text{osc}})_{\pm}$ to the J_{osc} and H_{osc} in Ref. [3].

Consider next the metric

$$(3.8) \qquad d\vec{x}^2 + 2dt\left[ds + \tfrac{1}{2}\epsilon_{ij}\mathcal{B}x^j dx^i\right]$$

where $\vec{x} \in \mathbf{R}^2$ and $\mathcal{B}$ is a constant. Its null geodesics describe a charged particle in a uniform magnetic field in the plane [6]. Again, when imposing equivariance, Eq. (2.1) reduces precisely to Eq. (1.1) with $\Lambda = \lambda/2$ and covariant derivative D_ω given as in Eq. (1.2). The metric (3.8) is readily transformed into an oscillator metric (3.1): the mapping $\varphi(t, \vec{x}, s) = (t_{\text{osc}}, \vec{x}_{\text{osc}}, s_{\text{osc}})$ given by

$$(3.9) \qquad t_{\text{osc}} = t, \qquad x^i_{\text{osc}} = x^i \cos\omega t + \epsilon^i_j x^j \sin\omega t, \qquad s_{\text{osc}} = s$$

— which amounts to switching to a rotating frame with angular velocity $\omega = \mathcal{B}/2$ — takes the 'constant $\mathcal{B}$-metric' (3.8) into the oscillator metric (3.1). The vertical vectors $\partial_{s_{\text{osc}}}$ and ∂_s are permuted. Thus, the time-dependent rotation (3.9) followed by the transformation (3.3), which projects to the coordinate transformation (1.5) of Refs. [2] and [3], carries conformally the constant-$\mathcal{B}$ metric (3.8) into the $\omega = 0$-metric. It carries therefore the 'empty' space solution $e^{is}\Psi$ with Ψ as in (2.7) into that in a uniform magnetic field background according to Eq. (2.10). Taking into account the equivariance, we get the formulæ of [2] i.e. (1.4) without the $\mathcal{N}$-terms — but multiplied with the Maslov factor $(-1)^j$. (The $\mathcal{N}$-term arises due to a subsequent gauge transformation required by the gauge fixing in [3]).

It also allows to 'export' the Schrödinger symmetry to non-relativistic Chern-Simons theory in the constant magnetic field background. The [rather complicated] generators, listed in Ref. [11], are readily obtained using Eq. (2.12). For example, time-translation $t \to t + \tau$ in the $\mathcal{B}$-background amounts to a time translation for the oscillator with parameter τ plus a rotation with angle $\omega\tau$. Hence $H_{\mathcal{B}} = H_{\text{osc}} - \omega\mathcal{J} = \mathcal{H} + \omega^2\mathcal{K} - \omega\mathcal{J}$. Similarly, a space translation for $\mathcal{B}$ amounts, in 'empty' space, to a space translations and a boost, followed by a rotation, yielding $P^i_B = \mathcal{P}^i + \omega\,\epsilon^{ij}\mathcal{G}^j$, etc.

4. Conformally flat Bargmann spaces

All our preceding results apply to any Bargmann space which can be conformally mapped into Minkowski space in a ξ-preserving way. Now we describe these 'Schrödinger-conformally flat' spaces. In $D = n + 2 > 3$ dimensions, conformal flatness is guaranteed by the vanishing of the conformal Weyl tensor

$$(4.1) \qquad C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - \tfrac{4}{D-2}\,\delta^{[\mu}_{[\rho}R^{\nu]}{}_{\sigma]} + \tfrac{2}{(D-1)(D-2)}\,\delta^{[\mu}_{[\rho}\delta^{\nu]}_{\sigma]}R.$$

Now $R_{\mu\nu\rho\sigma}\xi^{\mu} \equiv 0$ for a Bargmann space, which implies some extra conditions on the curvature. Inserting the identity $\xi_{\mu}R^{\mu\nu}{}_{\rho\sigma} = 0$ into $C^{\mu\nu}{}_{\rho\sigma} = 0$, using the identity $\xi_{\mu}R^{\mu}_{\nu} \equiv 0$ ($R^{\nu}_{\sigma} \equiv R^{\mu\nu}{}_{\mu\sigma}$), we find $0 = -\left[\xi_{\rho}R^{\nu}_{\sigma} - \xi_{\sigma}R^{\nu}_{\rho}\right] + R/(D-1)\left[\xi_{\rho}\delta^{\nu}_{\sigma} - \xi_{\sigma}\delta^{\nu}_{\rho}\right]$. Contracting again with ξ^{σ} and using that ξ is null, we end up with $R\,\xi_{\rho}\xi^{\nu} = 0$. Hence the scalar curvature vanishes, $R = 0$. Then the previous equation yields $\xi_{[\rho}R^{\nu}_{\sigma]} = 0$ and thus $R^{\nu}_{\sigma} = \xi_{\sigma}\eta^{\nu}$ for some vector field η. Using the symmetry of the Ricci tensor, $R_{[\mu\nu]} = 0$, we find that $\eta = \varrho\,\xi$ for some function ϱ. We finally get the *consistency relation*

$$(4.2) \qquad\qquad R_{\mu\nu} = \varrho\,\xi_{\mu}\xi_{\nu}.$$

The Bianchi identities ($\nabla_{\mu}R^{\mu}_{\nu} = 0$ since $R = 0$) yield $\xi^{\mu}\partial_{\mu}\varrho = 0$, i.e. ϱ is a function on spacetime Q. The conformal Schrödinger-Weyl tensor is hence of the form

$$(4.3) \qquad\qquad C^{\mu\nu}{}_{\rho\sigma} = R^{\mu\nu}{}_{\rho\sigma} - \tfrac{4}{D-2}\,\varrho\,\delta^{[\mu}_{[\rho}\,\xi^{\nu]}\xi_{\sigma]}.$$

It is noteworthy that Eq. (4.2) is the Newton-Cartan field equation with $\varrho/(4\pi G)$ as the matter density of the sources.

It follows from Eq. (4.2) that the transverse Ricci tensor of a Schrödinger-conformal flat Bargmann metric necessarily vanishes, $R_{ij} = 0$ for each t. The transverse space is hence (locally) flat and we can choose $g_{ij} = g_{ij}(t)$. Then a change of coordinates $(t, \vec{x}, s) \to (t, G(t)^{1/2}\vec{x}, s)$ where $G(t)^i_j = g_{ij}(t)$ [which brings in a uniform magnetic field and/or an oscillator to the metric], casts our Bargmann metric into the form (2.8) with $g_{ij} = \delta_{ij}$ while ξ remains unchanged.

The non-zero components of the Weyl tensor of the general $D = 4$ Brinkmann metric (2.8) are found as

$$C_{xyxt} = -C_{ytts} = -\tfrac{1}{4}\partial_x\mathcal{B},$$

$$C_{xyyt} = +C_{xtts} = -\tfrac{1}{4}\partial_y\mathcal{B},$$

$$(4.4) \qquad C_{xtxt} = -\tfrac{1}{2}\left[\partial_t(\partial_y\mathcal{A}_y - \partial_x\mathcal{A}_x) - \mathcal{A}_x\partial_y\mathcal{B},\right] + \tfrac{1}{2}\left[\partial_x^2 - \partial_y^2\right]U,$$

$$C_{ytyt} = +\tfrac{1}{2}\left[\partial_t(\partial_y\mathcal{A}_y - \partial_x\mathcal{A}_x) - \mathcal{A}_y\partial_x\mathcal{B}\right] - \tfrac{1}{2}\left[\partial_x^2 - \partial_y^2\right]U,$$

$$C_{xtyt} = +\tfrac{1}{2}\left[\partial_t(\partial_x\mathcal{A}_y + \partial_y\mathcal{A}_x) + 2\partial_x\partial_yU\right] - \tfrac{1}{4}(\mathcal{A}_x\partial_x - \mathcal{A}_y\partial_y)\mathcal{B}.$$

Then Schrödinger-conformal flatness requires

$$(4.5) \qquad \begin{cases} \mathcal{A}_i = \tfrac{1}{2}\epsilon_{ij}\mathcal{B}(t)x^j + a_i, & \vec{\nabla}\times\vec{a} = 0, & \partial_t\vec{a} = 0, \\[2mm] U(t, \vec{x}) = \tfrac{1}{2}C(t)r^2 + \vec{F}(t)\cdot\vec{x} + K(t). \end{cases}$$

(Note, *en passant*, that (4.2) automatically holds: the only non-vanishing component of the Ricci tensor is $R_{tt} = -\partial_t(\vec{\nabla}\cdot\vec{\mathcal{A}}) - \tfrac{1}{2}\mathcal{B}^2 - \Delta U$).

This Schrödinger-conformally flat metric hence allows one to describe a uniform magnetic field $\mathcal{B}(t)$, an attractive $[C(t) = \omega^2(t)]$ or repulsive $[C(t) = -\omega^2(t)]$ isotropic oscillator and a uniform force field $\vec{F}(t)$ in the plane which may all depend arbitrarily on time. It also includes a curlfree vector potential $\vec{a}(\vec{x})$ that can be gauged away if the transverse space is simply connected: $a_i = \partial_i f$ and the coordinate transformation $(t, \vec{x}, s) \to (t, \vec{x}, s + f)$ results in the 'gauge' transformation $\mathcal{A}_i \to \mathcal{A}_i - \partial_i f = -\frac{1}{2}\mathcal{B} \epsilon_{ij} x^j$. However, if space is not simply connected, we can also include an external Aharonov-Bohm-type vector potential.

Being conformally related, all these metrics share the symmetries of flat Bargmann space: for example, if the transverse space is $\mathbf{R}^2$ we get the full Schrödinger symmetry; for $\mathbf{R}^2 \setminus \{0\}$ the symmetry is reduced rather to $o(2) \times o(2,1) \times \mathbf{R}$, just like for a magnetic vortex [12].

The case of a constant electric field is quite amusing. Its metric, $d\vec{x}^2 + 2dtds - 2\vec{F}\cdot\vec{x}dt^2$, can be brought to the free form by switching to an accelerated coordinate system,

(4.6)
$$\vec{X} = \vec{x} + \tfrac{1}{2}\vec{F}\,t^2, \qquad T = t, \qquad S = s - \vec{F}\cdot\vec{x}\,t - \tfrac{1}{6}\vec{F}^2 t^3.$$

This example (chosen by Einstein to illustrate the equivalence principle) also shows that the action of the Schrödinger group — e.g. a rotation — looks quite different in the inertial and in the moving frames.

Let us finally mention that the above results admit a 'gauge theoretic' interpretation. In conformal (Lorentz) geometry, the Weyl tensor $C_{\mu\nu\rho\sigma}$ arises as part of the $o(n+2,2)$-valued curvature of a Cartan connection for a $D = n + 2$ dimensional base manifold. The Schrödinger-conformal geometry in this dimension can be viewed as a reduction of the standard conformal geometry to the Schrödinger subgroup $\mathrm{Sch}(n+1,1) \subset \mathrm{O}(n+2,2)$ [13]. The curvature of the reduced Cartan connection then defines the Schrödinger-Weyl tensor which is thus characterized by

(4.7)
$$C_{\mu\nu\rho\sigma}\,\xi^\mu = 0,$$

a property coming from the previous embedding and strictly equivalent to Eqs. (4.2,3).

5. Conclusion

Our 'non-relativistic Kaluza-Klein' approach provides a unified view on the various vortex constructions in external fields, explains the common origin of the large symmetries, and allows us to describe all such spaces. We have also pointed out that the formula (1.4) may require a slight modification for times larger then a half oscillator-period.

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